LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

corrected version (15 feb 2016)

Macroeconomics II

Calculation of growth rates: a note

by José António Pereirinha (15 feb 2016)

1. Discrete time-path of a variable: discrete growth rate

Let X be a variable with economic meaning, which assumes positive (and only positive) values that may change through time.

Let time be represented by a series of periods: 0, 1, ..., T. Then, we can describe the time path of this variable by a discrete-time model. The series

(1) $x_0, x_1, ..., x_T$

describe the discrete-time path of variable X.

The discrete growth rate of this variable in period t (in relation to t-1) is the value r_t such that:

(2) $x_t = x_{t-1} . (1+r_t), t = 0, ..., T$

what means that:

(3) $\boxed{r_t = (x_t / x_{t-1}) - 1 = (x_t - x_{t-1})/x_{t-1} = \Delta x_t/x_{t-1}}$

The value of r_t can be positive, null or negative.

2. Average growth rate (discrete)

Let us consider the whole period from t=0 up to t=T.

If r remains constant through time, that is, $r_t = r$, for t = 0, 1, ..., T, then:

(4) $x_t = x_{t-1}.(1+r)$ for all t = 0, ..., T

which implies:

(5) $x_T = (1+r)^T.x_0$

But this is unusual. The growth rates $r_1, r_2, ..., r_T$ may be different, and we may be interested to know the average growth rate in the whole period from 0 to T. The average growth rate is the rate r* such that:

(6) $x_T = (1+r^*)^T.x_0$

From (6) above we get:

(7) $\ln x_T = T. \ln (1+r^*) + \ln x_0$

$\ln(1+r^*) = (\ln x_T - \ln x_0)/T$

$\ln(1+r^*) = \ln(x_T/x_0)/T$

$1+r^* = (x_T/x_0)^{1/T}$

$\boxed{r^* = (x_T/x_0)^{1/T} - 1}$

It is possible to prove that r* is an average of the growth rates $r_1, r_2, ..., r_T$. But it is not the arithmetic average of such rates. Instead, fom (7) we get:

(8) $r^* = (x_T/x_0)^{1/T} - 1 = (x_T/x_{T-1} \,.\, x_{T-1}/x_{T-2} \,.\, ... \, x_2/x_1 \,.\, x_1/x_0)^{1/T} - 1$

We may then conclude that although the average growth rate for the whole period may be calculated without information on the growth rates in the intermediate periods (see (7)), it may be expressed as the geometric average of these growth rates (see (8)).

3. Continuous-time path of a variable: instantaneous growth rate

We may describe the time-path of the variable X using a continuous-time model. That is, we may assume that time flows continuously, and X assumes values in each moment of the whole period. Let x(t) be a continuous function of t, with a continuous first derivative in relation to t, for all t from 0 to T.

The instantaneous growth rate of X at moment t is

(9) $\boxed{r(t) = (d\,x(t)/d\,t)/x(t)}$

that compares to (3) in the discrete-time path case.

4. How to calculate the instantaneous growth rate

From (9) we get:

(10) $r(t) = (d\,x(t)/d\,t)/x(t) = d\ln x(t)/d\,t$

Let r(t) = r, a constant through the whole period. Then, from (9) we get:

(11) $d\ln x(t)/d\,t = r$

$\ln x(t) = r.t + \text{const.}$

$x(t) = e^{r.t} \,.\, \text{const} \ (= X(0))$

and then:

(12) $\boxed{X(t) = X(0) \,.\, e^{r.t}}$

where r is the instantaneous growth rate of the variable X, assuming that it has a continuous-time path through time, x(t) and also assuming that r(t) = r, a constant.

corrected version (15 feb 2016)

5. Average instantaneous growth rate (continuous)

But the general case is that the growth rate may vary through time, that is, r(t). If r(t) is not a constant, we may be interested to calculate the average instantaneous growth rate of the variable X throughout the whole period, from 0 to T. This is the value r* such that:

(13) $x(T) = x(0).e^{r^*.T}$

This rate may be calculated as follows. From (13) we get:

(14) $\ln x(T) = \ln x(0) + r^*.T$

$r^* = (\ln x(t) - \ln x(0)) / T$

$\boxed{r^* = \ln (x(t)/x(0))^{1/T}}$

6. Discrete growth rates: properties

Let X and Y be two discrete variables and let r(x) and r(y) be the average growth rate in the whole period from 0 to T. Let K a constant in this whole period. We may prove that:

(15a) $r(K) = 0$

(15b) $r(K.x) = r(x)$

(15c) $r(x.y) = r(x) + r(y) + r(x).r(y)$

$= r(x) + r(y)$, for small values of r(x) and r(y)

(15d) $r(x/y) = (r(x) - r(y))/(1 + r(y))$

$= r(x) - r(y)$, for small values of r(x) and r(y)

(15e) $r(x^k) = (1 + r(x))^k - 1$

$= k . r(x)$, for small values of r(x)

The strategy of proof of these properties is quite straightforward. Let us prove (15c). In the period t we have x_t, y_t and, therefore, $x_t.y_t$. In the period t+1 we have x_{t+1}, y_{t+1} and $x_{t+1}.y_{t+1}$. Then we have:

(16) $r(x.y) = (x_{t+1}.y_{t+1} - x_t.y_t) / x_t.y_t =$

$= (y_{t+1}. x_{t+1} - y_{t+1}.x_t + y_{t+1}.x_t - x_t.y_t) / x_t.y_t =$

$= (y_{t+1}.(x_{t+1} - x_t) + x_t . (y_{t+1} - y_t)) / x_t.y_t =$

$= (y_{t+1}/y_t).r(x) + r(y) =$

$= (r(y) + 1) . r(x) + r(y) = r(x) + r(y) + r(x).r(y)$

7. Instantaneous growth rates: properties

Let X(t) and Y(t) be two continuous variables, and K a constant. Let r(x) be the average instantaneous growth rate of X in a period and r(y) be the average instantaneous growth rate in the same period. We may prove that:

(17a) r(K) = 0

(17b) r(K.x(t)) = r(x(t))

(17c) r(x(t).y(t)) = r(x(t)) + r(y(t))

(17d) r(1/x(t)) = - r(x(t))

(17e) r(x(t)/y(t)) = r(x(t)) - r(y(t))

(17f) $r(x(t)^K) = K.r(x(t))$

The strategy to prove such properties is quite straightforward. Let us prove (17c). If $x(t) = x(0).e^{r(x).t}$ and $y(t) = y(0).\ e^{r(y).t}$, then $x(t).y(t) = x(0).y(0).e^{(r(x)+r(y)).t}$. The average growth rate of x(t).y(t) is then r(x) + r(y).

8. The rule of 70 (extracted from W. Carlin & D. Soskice, 2015, page 273)

"The relationship between GDP per capita and time that is shown in the exponential growth model leads us to a neat rule. The rule allows us to calculate approximately how long it takes for GDP per capita to double if as economy is growing at a constant rate of growth:

(18a) doubling time = 70 / percentage growth rate

The rule of 70 comes from the fact that log 2 ≈ 0.7. If we take y0 to be the GDP per capita in period 0 and td to be the time it takes to double, then using the exponential growth formula, we have

(18b) $2.y_0 = y_0\ .\ \exp\ (g_y.t_d)$

We can take logs of both sides and rearrange to derive the rule of 70:

(18c) $t_d = \log 2/g_y$

(...) we saw that the trend rate of US GDP per capita growth was 2.4% in the post-war period. This leads to a doubling time of 20/2.4% = 29 years. If the rate of GDP per capita growth was faster than this, say 5%, then it would double roughly every 14 years. In contrast, a rate of growth at 1% would see GDP per capita taking 70 years to double"

9. Variables in macroeconomic models: stocks and flows

In the growth models that will be studied in Macro 2, the aggregate production function plays an important role. It is described as:

(19) $Y_t = F(K_t, L_t, A_t)$

where Y_t is output (GDP), K_t is capital input, L_t is labour input and A_t is the "level of technology". The time index t may refer to a point in time (31st December 2015, for example), or to a time interval [t, t+1) (for example "year 2015", which is defined in the time interval 1st January 2015, 31st December 2015). It depends on the context. The above production function is an example of that. Output is a flow variable: the index t refers to a time interval ("year", in general), while capital input and labour input refers to a point in time (31st December of that year). The accumulation of the stock of capital is described as:

(20a) $K_{t+1} - K_t = I_t - \delta.K_t$

where I denotes investment and δ is the rate of depreciation of capital, $0 \leq \delta \leq 1$. In continuous time models the corresponding accumulation equation is:

(20b) $\dot{K}(t) = dK(t)/dt = I(t) - \delta.K(t)$

where the dot in $\dot{K}$ means the rate of change of K.

To be sure that you could understand, answer the following questions (from Christian Groth, February 2012):

a) At the theoretical level, what denominations (dimensions) should be attached to output, capital input, and labour input in a production function?

b) What is the denomination (dimension) attached to K in the accumulation equation?

c) Is there any consistency problem in the notation used in (19) vis-à-vis (20a)? Explain.

d) Suggest an interpretation that ensures that there is no consistency problem.

e) In continuous time we write aggregate (real) gross saving as $S(t) \equiv I(t)$: What is the denomination of S(t):

f) In continuous time, does the expression K(t) + S(t) make sense? Why or why not?

g) In discrete time, how can the expression $K_t + S_t$ be meaningfully interpreted?

corrected version (15 feb 2016)

10. Some conventions

In the context of the use of growth models (Solow-Swan, Romer, etc), only continuous (instantaneous) growth rates will be used. For this reason, and intending to make use of the conventional notation used in the literature for these models, we make the convention of, when using such models in the analysis, to denote the growth rate as "g". So, the growth rate of a variable X is denoted by g_x.

15 February 2016

José António Pereirinha